

High Order Schemes for CFD: A Review

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ABSTRACT

Over the past two decades there have been many research activities in the design and application of high order accurate numerical methods in computational fluid dynamics (CFD). High order methods are especially desirable for simulating flows with complicated solution structures. In this paper we give a review on the development and application of several classes of high order schemes in CFD, mainly concentrating on the simulation of compressible flows. An important feature of the compressible flow is the existence of shocks, interfaces and other discontinuities and often also complicated structure in the smooth part of the solution. This gives a unique challenge to the design of high order schemes to be non-oscillatory and yet still maintaining their high order accuracy. We will concentrate our discussion on the essentially non-oscillatory (ENO) and weighted ENO (WENO) finite difference and finite volume schemes, and discontinuous Galerkin (DG) finite element methods. We will attempt to describe their main properties and their relative strength and weakness. We will also briefly review their developments and applications, concentrating mainly on the results over the past five years.

Key Words: essentially non-oscillatory (ENO), weighted essentially non-oscillatory (WENO) , discontinuous Galerkin (DG), high order accuracy, finite difference, finite volume, finite element, computational fluid dynamics, compressible flow.

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1 Introduction

Over the past 20 to 30 years there have been a lot of research activities, both in the computational mathematics and in the physics and engineering communities, in designing, analyzing and applying high order accurate numerical methods for computational fluid dynamics (CFD), in particular for the simulation of compressible flows containing shocks and other discontinuities. In [154], Shu gave a review of several high order accurate non-oscillatory type schemes, including the weighted essentially non-oscillatory (WENO) finite difference schemes, WENO finite volume schemes, and discontinuous Galerkin (DG) finite element schemes, for their applications in CFD. This paper is a supplement of and an update on [154]. We will concentrate our discussion still on the essentially non-oscillatory (ENO) and WENO finite difference and finite volume schemes, and DG finite element methods. We will attempt to describe their main properties and their relative strength and weakness. We will also briefly review their developments and applications, concentrating mainly on the results over the past five years, after the publication of [154].

In this paper we refer to high order methods by those with an order of accuracy at least three, measured by local truncation errors where the solution is smooth. We should emphasize upfront, that for many application problems, adequate first and especially second order numerical methods are often good choices, when considering the balance among computational speed, simplicity of coding, and resolution required. This statement is especially valid if we seek solutions which are piecewise simple (almost linear) with several isolated discontinuities in between, for example, the solution of most Riemann problems. On the other hand, high order methods are good choices when we are concerned with situations containing both discontinuities and rich structures in the smooth part of the solutions, for example, the Rayleigh-Taylor instability simulation [195, 197], shock interaction with vortices [193, 194], and direct simulation of compressible turbulence [108, 162]. Therefore, the user should decide, based on the problem at hand, whether to use a high order scheme or to find a suitable first or second order scheme. We should however also emphasize that,

at least in certain situations, the solution structures may be so complicated and the time evolution of these structures so long that it is impractical to use low order methods to obtain an acceptable resolution. A good example is the evolution of a vortex given in [154]. When the vortex is convected for a long time, it is much more economical (in terms of operation count or CPU time) to use a high order scheme than a low order one to obtain the same level of resolution.

The content of this paper is as follows. In Section 2 we discuss briefly time discretization. In Section 3 we discuss finite difference and finite volume Eulerian WENO schemes. In Section 4 we give a short review of our recently developed finite volume Lagrangian ENO/WENO schemes. The DG method, again in an Eulerian formulation, is discussed in Section 5. We give some concluding remarks in Section 6.

2 Time discretizations

This review paper mainly discusses spatial discretization. Therefore we will give a short introduction on time discretization upfront. For compressible flow simulations, a very popular time discretization method is the class of total variation diminishing (TVD), also called strong stability preserving (SSP), Runge-Kutta and multistep methods.

The main advantage of this class of time discretizations is that they are convex combinations of first order forward Euler steps, hence they maintain strong stability properties in any semi-norm (total variation norm, maximum norm, entropy condition, etc.) of the forward Euler step. Thus one only needs to prove nonlinear stability for the first order forward Euler step, which is relatively easy in many situations (e.g. TVD schemes), and one automatically obtains the same strong stability property for the higher order time discretizations in this class.

These methods were first developed in [158] and [151], and later generalized in [55] and [56]. Many recent developments of these methods, including discussions on implicit schemes, can be found in the review paper [53]. The most popular scheme in this class is the following

third order Runge-Kutta method for solving

$$u_t = L(u, t)$$

where $L(u, t)$ is a (possibly nonlinear) spatial discretization operator:

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n, t^n) \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}, t^n + \Delta t) \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}, t^n + \frac{1}{2}\Delta t). \end{aligned}$$

Discussions on other popular schemes in this class are documented in [53]. These include the discussion on high order schemes for linear problems [16, 56], development and analysis of good Runge-Kutta and multistep schemes in this class [48, 49, 52, 54, 65, 66, 74, 75, 80, 81, 143, 144], SSP hybrid methods [73], and SSP spectral deferred correction methods [98, 53]. The importance of using SSP time discretizations for CFD problems is addressed in [82, 53].

There is an alternative time discretization procedure, referred to as the Lax-Wendroff procedure, which starts with a Taylor expansion in time to the desired order of accuracy, and then with a replacement of time derivatives by spatial derivatives via a repeated usage of the PDE and its differentiated versions. The original finite volume ENO scheme [59] uses this time discretization procedure. The ADER schemes, e.g. [164], also belong to this category. An efficient Lax-Wendroff procedure for finite difference WENO schemes is developed in [130]. A Lax-Wendroff procedure for the DG method is developed in [126].

For problems containing significant diffusion or higher order spatial derivatives, such as Navier-Stokes equations for a Reynolds number which is not very high, or for a highly non-uniform mesh, an explicit time discretization might put a severe time step restriction for stability. Implicit time discretizations would then be in order. Several types of implicit time discretizations have been used in CFD and other applications, such as the implicit-explicit Runge-Kutta schemes which can treat different parts of the spatial operators by explicit and implicit discretization respectively, while still maintaining high order temporal accuracy

[79]. The spectral deferred correction method [45, 111] is also of such type. In [175], several different implicit and exponential type time discretization techniques have been compared, in the context of DG methods for PDEs with higher order spatial derivatives.

The Strang splitting procedure [160] is quite popular in many applications, especially for multi-dimensional problems, as it can reduce the approximation of a complicated PDE

$$u_t = A(u) + B(u)$$

where $A(u)$ and $B(u)$ are differential operators (e.g. $A(u)$ could be the derivative operator in x and $B(u)$ could be the derivative operator in y) to that of two simpler problems

$$u_t = A(u)$$

and

$$u_t = B(u)$$

approximated one after the other. This splitting is at least first order accurate and with a clever alternating procedure it can be upgraded to second order accuracy in time [160]. However, for general nonlinear problems, it is very difficult to obtain higher than second order temporal accuracy through such splitting procedure.

3 Finite difference and finite volume Eulerian WENO schemes

3.1 Finite difference WENO schemes

In this paper, a finite difference scheme refers to a scheme in which the numerical solution is represented by the *point values* u_i approximating that of the exact solution $u(x_i, t)$ at the grid point x_i . A *conservative* finite difference spatial discretization to a conservation law

$$u_t + f(u)_x = 0 \tag{3.1}$$

approximates the derivative $f(u)_x$ by a conservative difference

$$f(u)_x|_{x=x_j} \approx \frac{1}{\Delta x} \left(\hat{f}_{j+1/2} - \hat{f}_{j-1/2} \right)$$

where $\hat{f}_{j+1/2}$ is the numerical flux, which typically is a Lipschitz continuous function of several neighboring values u_i . In [159], a general procedure to form the numerical flux based on the point values of the solution to guarantee accuracy on uniform meshes is described, which forms the mathematical foundation for all higher than second order conservative finite difference schemes. In this section we are mainly concerned with the high order WENO finite difference scheme developed in [77]. For the simplest case of a scalar equation (3.1) and if $f'(u) \geq 0$, the fifth order finite difference WENO scheme in [77] has the numerical flux given by

$$\hat{f}_{j+1/2} = w_1 \hat{f}_{j+1/2}^{(1)} + w_2 \hat{f}_{j+1/2}^{(2)} + w_3 \hat{f}_{j+1/2}^{(3)}$$

where $\hat{f}_{j+1/2}^{(i)}$ are three third order fluxes on three different stencils given by

$$\begin{aligned} \hat{f}_{j+1/2}^{(1)} &= \frac{1}{3}f(u_{j-2}) - \frac{7}{6}f(u_{j-1}) + \frac{11}{6}f(u_j), \\ \hat{f}_{j+1/2}^{(2)} &= -\frac{1}{6}f(u_{j-1}) + \frac{5}{6}f(u_j) + \frac{1}{3}f(u_{j+1}), \\ \hat{f}_{j+1/2}^{(3)} &= \frac{1}{3}f(u_j) + \frac{5}{6}f(u_{j+1}) - \frac{1}{6}f(u_{j+2}), \end{aligned}$$

and the nonlinear weights w_i are given by

$$w_i = \frac{\tilde{w}_i}{\sum_{k=1}^3 \tilde{w}_k}, \quad \tilde{w}_k = \frac{\gamma_k}{(\varepsilon + \beta_k)^2},$$

with the linear weights γ_k given by

$$\gamma_1 = \frac{1}{10}, \quad \gamma_2 = \frac{3}{5}, \quad \gamma_3 = \frac{3}{10},$$

and the smoothness indicators β_k given by

$$\begin{aligned} \beta_1 &= \frac{13}{12} (f(u_{j-2}) - 2f(u_{j-1}) + f(u_j))^2 + \frac{1}{4} (f(u_{j-2}) - 4f(u_{j-1}) + 3f(u_j))^2 \\ \beta_2 &= \frac{13}{12} (f(u_{j-1}) - 2f(u_j) + f(u_{j+1}))^2 + \frac{1}{4} (f(u_{j-1}) - f(u_{j+1}))^2 \\ \beta_3 &= \frac{13}{12} (f(u_j) - 2f(u_{j+1}) + f(u_{j+2}))^2 + \frac{1}{4} (3f(u_j) - 4f(u_{j+1}) + f(u_{j+2}))^2. \end{aligned}$$

Finally, ε is a parameter to avoid the denominator to become 0 and is usually taken as $\varepsilon = 10^{-6}$ in the computation.

This finishes the description of the fifth order finite difference WENO scheme [77] in the simplest case. We refer to [77, 152, 153] for the mathematical derivation of the formulas given above. As we can see, the algorithm is actually quite simple to code and there is no tunable parameters in the scheme besides ε .

For scalar equations without the property $f'(u) \geq 0$, one could use a flux splitting

$$f(u) = f^+(u) + f^-(u), \quad \frac{df^+(u)}{du} \geq 0, \quad \frac{df^-(u)}{du} \leq 0,$$

and apply the procedure above to $f^+(u)$, and a mirror image (with respect to $j + 1/2$) procedure to $f^-(u)$. The only requirement for the splitting is that $f^\pm(u)$ should be as smooth functions of u as $f(u)$ is and as the order of the scheme requires (e.g. if the scheme is fifth order, $f(u)$ and $f^\pm(u)$ should all have five continuous derivatives with respect to u). In most applications the simple Lax-Friedrichs flux splitting

$$f^\pm(u) = \frac{1}{2}(f(u) \pm \alpha u), \quad \alpha = \max_u |f'(u)|$$

where the maximum is taken over the relevant range of u , is a good choice.

The main advantage of the finite difference framework is that multi-dimensional calculations do not increase the complexity of the algorithm. For example, if we would like to approximate the two-dimensional conservation law

$$u_t + f(u)_x + g(u)_y = 0,$$

we could simply take the one-dimensional data $v_i = u_{i,j}$ for fixed j , and the one-dimensional WENO approximation procedure described above, to obtain an approximation of $f(u)_x$ at the point (x_i, y_j) . Similarly, we can obtain the approximation to $g(u)_y$ by taking $w_j = u_{i,j}$ for fixed i and use the same one-dimensional WENO approximation procedure described above. The total cost to compute the residue approximating $f(u)_x + g(u)_y$ is thus just twice that of the one-dimensional evaluation of $f(u)_x$ for each grid point. The code is also a simple generalization from one-dimension: essentially we only need to add an outside “do-loop” to

the one-dimensional code to get the two dimensional code. For this reason, when WENO schemes are used in applications, the finite difference version is the top choice for most cases.

For systems of hyperbolic conservation laws, the nonlinear part of the WENO procedure (i.e. the determination of the smoothness indicators β_k and hence the nonlinear weights w_i) should be carried out in local characteristic fields. In this procedure, we would first find an average $u_{j+1/2}$ of u_j and u_{j+1} (e.g. the Roe average [141] which exists for many physical systems), and compute the left and right eigenvectors of the Jacobian $f'(u_{j+1/2})$ and put them into the rows of the matrix $R_{j+1/2}^{-1}$ and the columns of the matrix $R_{j+1/2}$, respectively, such that $R_{j+1/2}^{-1} f'(u_{j+1/2}) R_{j+1/2} = \Lambda_{j+1/2}$ where $\Lambda_{j+1/2}$ is a diagonal matrix containing the real eigenvalues of $f'(u_{j+1/2})$. We then transform all the quantities needed for evaluating the numerical flux $\hat{f}_{j+1/2}$ to the local characteristic fields by left multiplying them with $R_{j+1/2}^{-1}$, and then computes the numerical fluxes by the scalar procedure in each characteristic field. Finally, the flux in the original physical space is obtained by left multiplying the numerical flux obtained in the local characteristic fields with $R_{j+1/2}$. This characteristic decomposition procedure is very costly, because of the many matrix-vector multiplications, and it accounts for the major additional CPU cost of high order WENO schemes versus lower order TVD schemes. There are some applications for which this characteristic decomposition is not necessary and the less expensive component by component WENO approximation would do just fine, however for the more demanding numerical tests involving strong shocks and for higher order WENO schemes, the characteristic decomposition procedure is often necessary, see, e.g. [129] for a comparison of characteristic versus component versions of WENO schemes in the context of central scheme framework.

One important restriction of conservative finite difference WENO schemes, or of any conservative finite difference schemes which are third or higher order of accuracy, is that they can only be applied to uniform or smooth non-uniform meshes. There is a theoretical barrier which prevents a finite difference scheme as formulated above from being both conservative and third (or higher) order accurate, see, e.g. [109]. Therefore, we can only use the finite

difference WENO scheme on uniform meshes or smooth curvilinear coordinates in multi-dimensional computation. If the application requires a complicated geometry or adaptive mesh which cannot be accommodated by smooth curvilinear coordinates, then we can use the multi-domain finite difference WENO method in [147] as long as the computational domain can be covered by a union of (not necessarily non-overlapping) simple domains in which a smooth curvilinear coordinate can be defined. We must emphasize, however, that the method in [147] is not conservative, although the conservation error is small and the numerical scheme still converges to weak solutions of the PDE under reasonable assumptions [147]. Another attempt to overcome this restriction is through the framework of residual distribution type finite difference WENO schemes, which have been designed in [26, 27] for steady state conservation laws and convection dominated convection diffusion equations. These schemes can be used on arbitrary *non-smooth* tensor product curvilinear meshes, are conservative, and have the same cost as the usual conservative finite difference schemes for two dimensional problems. However, the schemes in [26, 27] are only for steady state calculations. The design of efficient counter-part for time accurate problems is still an ongoing work.

In most of the applications of WENO schemes, the fifth order finite difference WENO procedure is used. However, the algorithm construction procedure is quite general, WENO finite difference schemes of different orders of accuracy have been documented in, e.g. [5].

We remark that the choice of smoothness indicators and nonlinear weights in the WENO scheme is not unique. The choice in [77] is based on extensive numerical experiments when considering the balance of efficiency, simplicity, accuracy and non-oscillatory performance. For an improvement of the choice of nonlinear weights to enhance accuracy, see [62]. For another set of smoothness indicators suitable for steady state calculation, see [192].

The WENO schemes with suitable choice of smoothness indicators and nonlinear weights can be proved to be uniformly high order accurate in smooth regions of the solution, including at smooth extrema. This is verified numerically and it is the main difference between high

order WENO schemes and total variation diminishing (TVD) schemes. Near discontinuities, WENO schemes behave very similarly to an ENO scheme [59, 158, 159] in terms of non-oscillatory performance. The solution has a sharp and non-oscillatory discontinuity transition even for strong shocks. The numerical flux has the same smoothness dependency on its arguments as that of the physical flux $f(u)$. This helps in a convergence analysis for smooth solutions [77] and in steady state convergence [192].

An important feature of the WENO scheme is that the approximation is self similar. That is, when fully discretized with the Runge-Kutta methods in Section 2, the scheme is invariant when the spatial and time variables are scaled by the same factor. This is because the time step Δt and spatial mesh size Δx do not appear individually in the scheme, only their ratio $\Delta t/\Delta x$ which is related to the CFL number appears. This property is very important for applications.

Although WENO schemes perform very well in applications, the mathematical theory is not abundant. For smooth solutions, the WENO schemes would converge with the correct order of accuracy, as proved in [77]. For general discontinuous solutions, there is a recent result in [184] about the L^2 stability of the numerical solution of third order WENO schemes.

Recently, there have been activities in designing compact WENO schemes, with the objective of obtaining WENO schemes which have the advantage of the usual compact schemes in wave resolution, see for example [40, 191].

One difficulty of numerical solutions for conservation laws is that contact discontinuities are much more difficult to resolve sharply than shocks. This is because the characteristics are parallel to the discontinuity for a contact discontinuity, while they converge into the discontinuity for a shock. Therefore, a numerical shock is usually stable with a fixed number of transition points for long time simulation, while a numerical contact discontinuity may progressively become more smeared for longer time simulation. Recently, in [180], Xu and Shu designed the anti-diffusive flux correction WENO method based on the first order anti-diffusive flux correction in [41]. The anti-diffusive WENO schemes in [180] can significantly

improve the sharpness of contact discontinuities without affecting the accuracy and stability of the scheme for other parts of the solution. In [182], the anti-diffusive flux corrected WENO scheme is used to simulate the transport of pollutant, with very good resolution.

We summarize the pros and cons of finite difference WENO schemes below:

1. Finite difference WENO schemes can be used in a computational domain that can be covered either by uniform or by smooth curvilinear meshes. The smoothness of the mesh must be comparable with the order of accuracy of the scheme in order to obtain a truly high order result.
2. If the computational problem does allow for such meshes, then the finite difference WENO schemes are good choices as they are easy to code and fast to compute, especially for multi-dimensional problems. Usually the fifth order WENO scheme is the best choice, unless the nature of the problem asks for higher orders of resolution.
3. The finite difference WENO schemes can also be used in an adaptive mesh environment, provided that a smooth (in space and time) mesh can be generated. To generate such meshes is not easy, especially for higher order schemes where the requirement for the smoothness of the meshes is stronger.

3.2 Finite volume WENO schemes

A finite volume scheme for a conservation law such as (3.1) approximates its integral version.

If we integrate the PDE (3.1) over the interval $I_j = (x_{j-1/2}, x_{j+1/2})$, we obtain

$$\frac{d}{dt} \int_{I_j} u(x, t) dx + f(u(x_{j+1/2}, t)) - f(u(x_{j-1/2}, t)) = 0. \quad (3.2)$$

The numerical solution for a finite volume scheme is represented by the *cell averages* \bar{u}_j approximating that of the exact solution $\frac{1}{\Delta x_j} \int_{I_j} u(x, t) dx$ where $\Delta x_j = x_{j+1/2} - x_{j-1/2}$ is the mesh size of the cell I_j . If we would like to convert (3.2) into a scheme, we would need to obtain a good approximation for the point values $u(x_{j+1/2}, t)$ for all j , based on

the information of the cell averages \bar{u}_j . For stability, the term $f(u(x_{j+1/2}, t))$ in (3.2) is approximated by

$$f(u(x_{j+1/2}, t)) \approx \hat{h}(u_{j+1/2}^-, u_{j+1/2}^+)$$

where $\hat{h}(u^-, u^+)$ is a monotone flux satisfying the following properties

1. $\hat{h}(u^-, u^+)$ is non-decreasing in its first argument u^- and non-increasing in its second argument u^+ , symbolically $\hat{h}(\uparrow, \downarrow)$;
2. $\hat{h}(u^-, u^+)$ is consistent with the physical flux $f(u)$, i.e. $\hat{h}(u, u) = f(u)$;
3. $\hat{h}(u^-, u^+)$ is Lipschitz continuous with respect to both arguments u^- and u^+ .

Examples of monotone fluxes can be found in, e.g. [90]. Now the finite volume scheme becomes

$$\frac{d}{dt} \bar{u}_j + \frac{1}{\Delta x_j} \left(\hat{h}(u_{j+1/2}^-, u_{j+1/2}^+) - \hat{h}(u_{j-1/2}^-, u_{j-1/2}^+) \right) = 0 \quad (3.3)$$

which can be discretized in time by, e.g. the TVD Runge-Kutta method described in Section 2. Therefore, the main ingredient of a finite volume scheme is the reconstruction, to obtain accurate and non-oscillatory approximations $u_{j+1/2}^\pm$ from the cell averages \bar{u}_j . It is in this reconstruction procedure that WENO is playing a crucial role.

Let us explain the fifth order WENO reconstruction to $u_{j+1/2}^-$ as an example. The superscript “-” refers to the fact that the reconstruction has a stencil which is biased to the left with respect to the target location $x_{j+1/2}$. For the fifth order reconstruction, which needs 5 cells in the stencil, such a biased stencil consists of I_{j-2} , I_{j-1} , I_j , I_{j+1} and I_{j+2} , namely 3 cells to the left of $x_{j+1/2}$ and 2 cells to the right of it. The fifth order WENO reconstruction is given by

$$u_{j+1/2}^- = w_1 u_{j+1/2}^{(1)} + w_2 u_{j+1/2}^{(2)} + w_3 u_{j+1/2}^{(3)}$$

where $u_{j+1/2}^{(i)}$ are three third order reconstructions based on three different smaller stencils given by

$$\begin{aligned} u_{j+1/2}^{(1)} &= \frac{1}{3}\bar{u}_{j-2} - \frac{7}{6}\bar{u}_{j-1} + \frac{11}{6}\bar{u}_j, \\ u_{j+1/2}^{(2)} &= -\frac{1}{6}\bar{u}_{j-1} + \frac{5}{6}\bar{u}_j + \frac{1}{3}\bar{u}_{j+1}, \\ u_{j+1/2}^{(3)} &= \frac{1}{3}\bar{u}_j + \frac{5}{6}\bar{u}_{j+1} - \frac{1}{6}\bar{u}_{j+2}, \end{aligned}$$

and the nonlinear weights w_i are given by

$$w_i = \frac{\tilde{w}_i}{\sum_{k=1}^3 \tilde{w}_k}, \quad \tilde{w}_k = \frac{\gamma_k}{(\varepsilon + \beta_k)^2},$$

with the linear weights γ_k given by

$$\gamma_1 = \frac{1}{10}, \quad \gamma_2 = \frac{3}{5}, \quad \gamma_3 = \frac{3}{10},$$

and the smoothness indicators β_k given by

$$\begin{aligned} \beta_1 &= \frac{13}{12}(\bar{u}_{j-2} - 2\bar{u}_{j-1} + \bar{u}_j)^2 + \frac{1}{4}(\bar{u}_{j-2} - 4\bar{u}_{j-1} + 3\bar{u}_j)^2 \\ \beta_2 &= \frac{13}{12}(\bar{u}_{j-1} - 2\bar{u}_j + \bar{u}_{j+1})^2 + \frac{1}{4}(\bar{u}_{j-1} - \bar{u}_{j+1})^2 \\ \beta_3 &= \frac{13}{12}(\bar{u}_j - 2\bar{u}_{j+1} + \bar{u}_{j+2})^2 + \frac{1}{4}(3\bar{u}_j - 4\bar{u}_{j+1} + \bar{u}_{j+2})^2. \end{aligned}$$

Finally, as before, ε is a parameter to avoid the denominator to become 0 and is usually taken as $\varepsilon = 10^{-6}$ in the computation.

The reader will have noticed the striking similarity between the reconstruction procedure described here and the one for computing finite difference WENO fluxes in the previous subsection. In fact, the two procedures are identical once we replace the point values of the physical fluxes $f(u_j)$ in the finite difference flux evaluation by the cell averages \bar{u}_j in the finite volume reconstruction procedure. This simple mathematical fact was first observed in [159] and it forms the mathematical justification for the construction of high order conservative finite difference schemes.

A major difference between the finite volume schemes and finite difference schemes is that the reconstruction procedure for a finite volume scheme does not require the mesh to

be uniform or smooth. Even in the one dimensional case, this would make the finite volume schemes more flexible in applications. For multi-space dimensions, the finite volume scheme can be designed on arbitrary triangulations. The first finite volume WENO scheme, which is third order accurate in one space dimension, is designed in [97]. For finite volume WENO schemes on tensor product meshes, see for example [148]. For finite volume WENO schemes on arbitrary 2D and 3D triangulations, see for example [43, 44, 50, 71, 196].

We summarize the properties of finite volume WENO schemes and give a comparison with their finite difference counterparts.

1. In one space dimension, the finite volume WENO scheme and the finite difference WENO scheme of the same order of accuracy have the same complexity and cost, in fact they share the same WENO reconstruction procedure which is the major part of the algorithm. On the other hand, the finite volume WENO scheme is more flexible as it can be used on non-smooth meshes and arbitrary monotone fluxes. Therefore, in one space dimension the finite volume WENO scheme is a clear winner.
2. For multi-space dimensions, a major advantage of finite volume WENO schemes over finite difference WENO schemes is that the former can be designed for arbitrary triangulations. However, the finite volume WENO schemes (of third and higher orders of accuracy) are much more complicated to code and cost much more CPU time than the finite difference counterparts for the same order of accuracy. This is because they have to rely on multidimensional reconstructions (polynomials of 2 or 3 variables in 2D or 3D), and the flux integrals on the cell boundaries must be performed by multi-point Gaussian quadratures. As a rule of thumb, a finite volume WENO scheme is usually 4 times more expensive in 2D and 9 times more expensive in 3D, compared with a finite difference WENO scheme on the same mesh and of the same order of accuracy, see, e.g. [15] for such a comparison for ENO schemes. Therefore, if the geometry and the smoothness of the meshes allow, finite difference WENO schemes would be the winner for multi-space dimensions.

3. Finite volume WENO schemes should be used in the situation when it is impossible to apply a smooth curvilinear mesh. For example, the Lagrangian methods described in the next section.

3.3 Recent applications of WENO schemes in CFD

There are numerous applications of finite difference and finite volume WENO schemes in recent literature. In fact, the paper of Jiang and Shu [77], where the fifth order finite difference WENO scheme was developed, has now been cited 591 times (as of January 26, 2009) in the Web of Science (SCI) database, most of those cites are from application journals. The review paper [156] has listed some of these applications. In computational fluid dynamics, we could mention the following examples of applications:

- High speed and turbulent flows typically contain both strong shocks and complicated flow structures, making WENO schemes good simulation tools for them. Recent results include [22, 57, 89, 112, 123, 78, 163, 95, 157]. In particular, direct numerical simulation (DNS) of compressible turbulence is ideal to be simulated by high order WENO schemes, see for example [108, 138, 139, 162, 173, 174].
- Shock vortex interaction. This is an important physical process for sound generation. This problem has complicated solutions where shocks of various strengths and vorticity flows co-exist, making it ideal to be simulated by high order WENO schemes. Many papers in the literature use WENO schemes to study the details of shock vortex interactions in different physical setup and regimes. Recent examples include [122, 135, 193, 194].
- Reacting flows, detonations, and flames contain multiple scales in the solution together with shocks, making the numerical simulation of them very difficult. WENO schemes have been successfully used for such simulations. See for example [3, 61, 62, 63, 72, 146, 145].

- A recent application of shock mitigation and drag reduction by pulsed energy lines for supersonic flows past a wedge is studied in [85] using the multi-domain finite difference WENO scheme developed in [147]. The simulation results indicate a saving of energy (energy gained is larger than energy spent to generate the pulsed energy lines) for all studied configurations.
- Compressible multicomponent flows are simulated in [102, 107]. A pseudocompressibility method for the numerical simulation of incompressible multifluid flows is developed in [116]. Modeling and simulation for a particle-fluid two phase flow problem are studied in [190].
- Hydrodynamic and quasi-neutral approximations for collisionless two-species plasmas are performed in [87]. Effects of shock waves on Rayleigh-Taylor instability are studied in [197].
- Shallow water equations and sediment transport equations are studied in [37]. For shallow water equations, well-balanced WENO schemes for both still water and moving water have been designed [114, 115, 170, 176, 177, 178, 179].
- Numerical simulation of interphase mass transfer with the level set approach is performed in [185]. Migrating and dissolving liquid drops are studied in [6, 7]. For flows in liquid helium, nonlinear effects and shock formation in the focusing of a spherical acoustic wave are studied in [4].
- A careful numerical study is performed in [149] for Euler equations and in [195] for Navier-Stokes equations to demonstrate that for flow problems containing both shocks and complicated smooth region structures, it is more economical to use high order WENO schemes than to use lower order schemes to achieve the same level of resolution. See also [188] for a comparison of schemes (including the WENO scheme) for the resolution of contact discontinuity layers, in which the WENO schemes performed

nicely.

- Even though not directly related to CFD, the WENO algorithm developed in [13, 14] can solve Boltzmann type equations for dimensions up to 5 (two space dimensions plus 3 phase space dimensions, plus time) with good resolution relative to Monte Carlo results on a single PC using a coarse mesh. This algorithm has been adapted to astrophysical / cosmology applications [124, 96, 125, 142]. It is expected that this algorithm would be useful for rarefied gas simulation and other situations when a Boltzmann formulation of the gas dynamics is necessary.

4 Finite volume Lagrangian ENO/WENO schemes

In this section we review a recent development of a class of finite volume Lagrangian ENO/WENO schemes for compressible Euler equations.

In numerical simulations of fluid flow, besides the Eulerian framework, in which the fluid flows through a grid fixed in space, there is another typical choice: the Lagrangian framework, in which the mesh moves with the local fluid velocity. More generally, the motion of the grid can also be chosen arbitrarily, then the method is called the Arbitrary Lagrangian-Eulerian method (ALE; cf. [67]). Comparing with Eulerian methods, Lagrangian methods and certain ALE methods can capture contact discontinuities sharply, thus they are widely used in many fields for multi-material flow simulations in CFD and other applications (e.g. in astrophysics).

In the past near six decades, many Lagrangian schemes (cf. [169, 9, 42, 12, 113, 106]) were developed which have made successes in simulating the multi-material flows. These schemes usually have first or second order accuracy in space and first order in time. More recently, Cheng and Shu started to investigate high order Lagrangian type schemes. A class of high order Lagrangian type schemes were proposed for solving the Euler equations both on the non-staggered quadrilateral meshes and the non-staggered curved quadrilateral meshes in [19, 21] respectively. The schemes are based on the high order ENO or simple

WENO reconstruction, are conservative for the density, momentum and total energy, and are essentially non-oscillatory. It can achieve at least uniformly second order accuracy on moving and distorted Lagrangian quadrilateral meshes and third order accuracy both in space and time on the curved quadrilateral meshes.

In the Lagrangian framework, the Euler equations in the following integral form is approximated,

$$\frac{d}{dt} \int_{\Omega(t)} \mathbf{U} d\Omega + \int_{\Gamma(t)} \mathbf{F} d\Gamma = 0 \quad (4.1)$$

where $\Omega(t)$ is the moving control volume enclosed by its boundary $\Gamma(t)$. The vector of the conserved variables \mathbf{U} and the flux vector \mathbf{F} are given by

$$\mathbf{U} = \begin{pmatrix} \rho \\ \mathbf{M} \\ E \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 \\ p \cdot \mathbf{n} \\ p\mathbf{u} \cdot \mathbf{n} \end{pmatrix}. \quad (4.2)$$

where ρ is the density, \mathbf{u} is the velocity, $\mathbf{M} = \rho\mathbf{u}$ is the momentum, E is the total energy and p is the pressure, \mathbf{n} denotes the unit outward normal to $\Gamma(t)$.

In [19], three steps are used to construct a high order ENO Lagrangian scheme, namely, spatial discretization, the determination of the vertex velocity and the time discretization. In the step of spatial discretization, to get a high order scheme in space, similarly to the Eulerian finite volume scheme, the high order ENO or WENO reconstruction procedure is used to obtain high order and non-oscillatory approximations to the solution at the Gaussian points along the cell boundary from the neighboring cell averages. The determination of velocity at the vertex is a special step for a Lagrangian scheme which decides how the mesh moves at the next time. The reconstruction information of density and momentum from its four neighboring cell is used in this step which guarantees the approximation of velocity also has the same order accuracy as the primitive conserved variables. In the step of time discretization, TVD Runge-Kutta type methods described in Section 2 are applied. One and two dimensional numerical examples both in the Cartesian and cylindrical coordinates are presented in [19] to demonstrate the performance of the schemes in terms of accuracy, resolution for discontinuities, and non-oscillatory properties.

In the accuracy test in [19], a phenomenon for the high order Lagrangian type scheme is observed, that is, the third order Lagrangian type scheme can only achieve second order accuracy on two dimensional distorted Lagrangian meshes. This is analyzed to be due to the error from the mesh approximation. Since in a Lagrangian simulation, a cell with an initially quadrilateral shape may not keep its shape as a quadrilateral at a later time. It usually becomes a curved quadrilateral. Thus if during the Lagrangian simulation the mesh is always kept as quadrilateral with straight-line edges, this approximation of the mesh will bring second order error into the scheme. So for a Lagrangian type scheme in multi-dimensions, it can be at most second order accurate if curved meshes are not used. In [21], Cheng and Shu demonstrate the previous claim by developing a third order scheme on curved quadrilateral meshes in two space dimensions. The reconstruction is based on the high order WENO procedure. Each curvilinear cell consists of four quadratically-curved edges by the information of the coordinates of its four vertices and the four middle points of its four edges. The accuracy test and some non-oscillatory tests are presented to verify the good performance of the scheme. The Lagrangian type scheme can also be extended to higher than third order accuracy if a higher order approximation is used on both the meshes and the discretization of the governing equations.

For the ALE method, usually it consists of three phases, a Lagrangian phase in which the solution and the grid are updated, a rezoning phase in which the nodes of the computational grid are moved to a more optimal position and a remapping phase in which the Lagrangian solution is transferred to the new grid. High order ALE method can also be obtained by using the high order Lagrangian type schemes in [19] and the high order ENO remapping method presented in [20]. Some numerical tests on this high order ALE method can be found in [19].

In general, comparing with the high order Eulerian schemes such as WENO schemes or DG schemes, the high order Lagrangian type scheme can capture contact discontinuities sharply and have higher resolution in the flow field. However, the robustness of the high

order Lagrangian type schemes is still not satisfactory which is mainly due to the distortion of meshes. Better approaches to control the movements of meshes within the expected order accuracy and high order Lagrangian type schemes or ALE schemes for the multi-material flows are still being investigated.

5 Discontinuous Galerkin methods

Similar to a finite volume scheme, a discontinuous Galerkin (DG) method for a conservation law such as (3.1) also approximates an integral version of it. Instead of integrating the PDE (3.1) over the interval $I_j = (x_{j-1/2}, x_{j+1/2})$ directly, we first multiply it by a test function $v(x)$, then integrate over the interval $I_j = (x_{j-1/2}, x_{j+1/2})$, and integrate by parts to obtain

$$\int_{I_j} u_t(x, t)v(x)dx - \int_{I_j} f(u(x, t))v_x(x)dx + f(u(x_{j+1/2}, t))v(x_{j+1/2}) - f(u(x_{j-1/2}, t))v(x_{j-1/2}) = 0. \quad (5.1)$$

The numerical solution for a DG scheme is represented by a piecewise polynomial function, still denoted as $u(x, t)$, which is a polynomial of degree k in each cell I_j . The test function $v(x)$ is also taken as a piecewise polynomial of degree k . If we would like to convert (5.1) into a scheme, we would only need to interpret $f(u(x_{j+1/2}, t))$, $v(x_{j+1/2})$, $f(u(x_{j-1/2}, t))$ and $v(x_{j-1/2})$ adequately, as both the solution $u(x, t)$ and the test function $v(x)$ are discontinuous at the interface $x_{j+1/2}$. Based on the similarity with finite volume schemes, these interface terms are handled as follows:

1. Replace $f(u(x_{j+1/2}, t))$ by a monotone flux $\hat{h}(u(x_{j+1/2}^-, t), u(x_{j+1/2}^+, t))$. Of course $f(u(x_{j-1/2}, t))$ is also replaced by $\hat{h}(u(x_{j-1/2}^-, t), u(x_{j-1/2}^+, t))$.
2. Replace the test function values at the interface by its values taken from inside the cell I_j , namely replace $v(x_{j+1/2})$ by $v(x_{j+1/2}^-)$ and replace $v(x_{j-1/2})$ by $v(x_{j-1/2}^+)$.

The DG scheme is thus obtained: we would like to find a piecewise polynomial of degree k , still denoted by $u(x, t)$, such that the following equality holds for all test functions $v(x)$

which are piecewise polynomials of degree k :

$$\int_{I_j} u_t(x, t)v(x)dx - \int_{I_j} f(u(x, t))v_x(x)dx + \hat{h}(u(x_{j+1/2}^-, t), u(x_{j+1/2}^+, t))v(x_{j+1/2}^-) - \hat{h}(u(x_{j-1/2}^-, t), u(x_{j-1/2}^+, t))v(x_{j-1/2}^+) = 0. \quad (5.2)$$

After choosing a suitable basis for piecewise polynomials of degree k , the method of the lines ODE (5.2) can again be discretized in time by, e.g. the TVD Runge-Kutta method described in Section 2. By choosing a locally orthogonal basis of piecewise polynomials, or by inverting a small local mass matrix (which is only of the size $(k+1) \times (k+1)$ for piecewise polynomials of degree k), the resulting scheme is explicit, just like a finite difference or finite volume scheme discretized by explicit Runge-Kutta methods.

The monotone flux needed for the DG scheme is the same as the one used in finite volume schemes. A noticeable difference between the DG method and the finite volume scheme is that the former does not involve a reconstruction procedure, as the values needed to evaluate the flux $\hat{h}(u(x_{j+1/2}^-, t), u(x_{j+1/2}^+, t))$ are readily available, since $u(x, t)$ is a piecewise polynomial function which can be evaluated at any point. This would save a lot of computational cost since the reconstruction is the main computational intensive portion of a finite volume scheme. However, the price to pay is that the DG method would need to store and evolve more than one piece of information per cell: for a polynomial of degree k , a total of $k+1$ pieces of information (e.g. the coefficients a_0, a_1, \dots, a_k in the polynomial expression $a_0 + a_1x + \dots + a_kx^k$) must be stored and evolved per cell. This would certainly increase the memory requirement and computational time in the evolution stage of the algorithm. On balance, whether a finite volume scheme or a DG scheme is more cost-effective depends on the specific PDE, the complexity of the geometry and meshes, and the computer platform. A comparison of the relative efficiency between the finite volume and DG schemes is given in [198].

The DG method has the best provable stability property among all the methods discussed in this paper. One can prove a cell entropy inequality for the square entropy [76],

which implies L^2 stability for the full nonlinear scalar or symmetric system cases with discontinuous solutions, and any converged solution is an entropy solution for a convex scalar conservation law. This cell entropy inequality holds for all scalar nonlinear conservation laws and symmetric systems [76, 70], all orders of accuracy of the scheme, all space dimensions, arbitrary triangulation, and without the need to use the nonlinear limiters. No high order finite volume or finite difference schemes have such comprehensive stability properties.

When applied to problems with smooth solutions or solutions containing only weak discontinuities, or when applied to linear PDEs such as the Maxwell equations or linearized Euler equations in aeroacoustics, the DG method as described above can be directly applied and will produce stable and accurate results. However, for nonlinear problems and solutions containing strong discontinuities, the unmodulated DG method is still subject to spurious oscillations, which may lead to nonlinear instability and blowup of the solution. This is because the L^2 stability is not strong enough to control the spurious oscillations in the numerical solution, and nonlinearity in the PDE would lead to interactions and possible amplifications of such spurious oscillations, leading to instability. Stability in a stronger norm such as the total variation semi-norm would be required for such situations, and this would need an additional stabilization realized by a nonlinear limiter. In earlier DG literature [32, 29, 33], the limiter adopted is the total variation bounded (TVB) limiter, originally designed in [150] for finite volume schemes. Such limiters are relatively simple to apply, and they can control spurious oscillations without destroying the accuracy of the scheme. Unfortunately, there is a parameter in such TVB limiters which must be tuned by the user. Other limiters have also been designed for DG schemes [10, 11]. In recent years, there is a series of work to apply the WENO methodology as a limiter for DG schemes. The general approach is to first identify the so-called ‘troubled cells’, namely those cells which might need the limiting procedure [86, 133]. Then, the solution polynomials in those troubled cells are replaced by reconstructed polynomials using the WENO methodology which maintain the original cell averages (conservation), have the same orders of accuracy as before, but are less oscillatory.

This approach looks very promising in combining the advantages of WENO schemes and DG methods. We refer to [132, 131, 134, 104, 199] for some of the representative work in this direction. In a recent work [1], the relationship between the DG method and the residual distribution method is explored, which may lead to an alternate way of designing limiters.

The first DG method was introduced in 1973 by Reed and Hill [136], in the framework of neutron transport, i.e. a time independent linear hyperbolic equation. It was later developed for solving nonlinear hyperbolic conservation laws containing first derivatives by Cockburn et al. in a series of papers [32, 31, 29, 33], in which the authors have established a framework to easily solve nonlinear time dependent problems, such as the Euler equations in compressible gas dynamics, using explicit, nonlinearly stable high order Runge-Kutta time discretizations described in Section 2 and DG discretization in space described above.

Because of the special choice of discontinuous, piecewise polynomials in the approximation space, the DG methods have certain flexibility and advantages, such as,

1. The DG method can be easily designed for any order of accuracy. In fact, the order of accuracy can be locally determined in each cell. This increase in order of accuracy does not rely on the complicated reconstruction procedure as in finite volume schemes.
2. It is easy for the DG method to handle complicated geometry and boundary conditions. They can be used on arbitrary triangulations, even those with hanging nodes. In principle, finite volume schemes can also handle complicated geometry, however the reconstruction procedure becomes quite involved [71, 196].
3. The DG method is extremely local in data communications. The evolution of the solution in each cell needs to communicate only with its immediate neighbors, regardless of the order of accuracy. The methods have excellent parallel efficiency, usually more than 99% for a fixed mesh, and more than 80% for a dynamic load balancing with adaptive meshes which change often during time evolution, see, e.g. [10, 137].
4. The DG method is flexible to h - p adaptivity. A very good example to illustrate the

capability of the DG method in h - p adaptivity, efficiency in parallel dynamic load balancing, and excellent resolution properties is the successful simulation of the Rayleigh-Taylor flow instabilities in [137].

The DG method can also be used on problems with second derivatives (diffusion terms such as those from the Navier-Stokes equations), see, e.g. [117, 34, 25]. One possible approach is the so-called local DG (LDG) methods [34]. Theoretical results about stability and rate of convergence are very similar to those for the first derivative PDEs. Unlike the traditional mixed method, such LDG methods for higher derivatives are truly local (the auxiliary variables introduced for the derivatives can be eliminated locally) and share with the DG method all the flexibility and advantages such as a tolerance of arbitrary triangulation with hanging nodes, parallel efficiency, easiness in h - p adaptivity, etc.

Recently, there has been a series of activities to design and analyze a class of central DG schemes, which use overlapping meshes and duplicative information (at each spatial point there are two finite element representations of the solution) to gain stability and other advantages. A class of hierarchical reconstruction limiters is also designed. See [99, 100, 101, 183].

The DG method framework can easily accommodate local approximation spaces with special properties, such as locally divergence-free or curl-free elements. This flexibility can be explored to design structure-preserving DG methods which can better approximate the relevant PDE with less computational cost. See for example the locally divergence-free DG method for the Maxwell equations [30] and for the MHD equations [92], and the locally curl-free DG method for the Hamilton-Jacobi equations [93] and for the multiscale modeling of dynamics of crystalline solids [171].

The DG method is suitable for certain multiscale applications. One approach is to use a domain decomposition, in which different models are used in different sub-domains. The compactness and the simple interface communication mechanism (via numerical fluxes) of the DG operator make such domain decomposition easy to design, see for example [17] in

which such a domain decomposition approach is tested on CFD and semiconductor device simulation applications. Another approach is to use the heterogeneous multiscale method (HMM) framework [46], see for example [18] in which such approach is explored for both linear and nonlinear CFD problems. The third approach is to use the flexibility of the DG method in choosing basis functions which are not polynomials [186], and use suitable multiscale basis functions obtained from asymptotic analysis to obtain accurate simulation results [187, 172].

For a detailed description of the DG method as well as its implementation and application, we refer to the lecture notes [28, 155]. The review paper [35] is also a good reference source. More recently, there are two special journal issues devoted to the DG method [36, 38], which contain many interesting papers in the development of the method in all aspects including algorithm design, analysis, implementation and applications. There are also several books published very recently [64, 91, 140] which cover various aspects of DG methods.

There are numerous applications of DG schemes in recent literature. Some of the applications can be found in the two special journal issues devoted to the DG method [36, 38], as well as in the books [64, 91, 140]. In computational fluid dynamics, we would like to mention the following examples of applications. We again concentrate on results in the past five years:

- Space-time DG method for Euler and Navier-Stokes equations have been developed. Such method provides the ultimate flexibility in adaptivity, as the mesh as well as the order of accuracy can be arbitrary in space and time. The methods in [83, 120, 121] are fully implicit in space-time, and an iterative Newton type solver must be applied for each time slab to obtain the numerical solution. When suitable techniques such as multigrid are used, the convergence of these iterative procedures is quite fast. The methods in [110, 118, 119] are locally implicit based on a special space-time mesh, hence can be solved more efficiently.
- A DG code for laminar and turbulent flows is developed in [88]. DG method is used to

solve steady-state compressible flows in [105]. Adaptive DG method for compressible Navier-Stokes equations is applied in [60]. High temperature equilibrium airflows are computed by the DG method in [167]. DG solution for the Reynolds-averaged Navier-Stokes and k - ω turbulence model is given in [8].

- DG methods for solving MHD equations are developed in [92, 161]. A DG method for the full two-fluid plasma model is discussed in [103]. DG methods for computational aeroacoustics in complex domains have been applied in [166].
- DG methods have been designed and applied to solve water wave equations. See, e.g. [2, 39, 51, 165, 170]. LDG methods for solving nonlinear porous medium equations are designed and tested in [168, 189].
- LDG schemes for solving model problems in phase transition theory are designed and analyzed in [58]. Applications of DG scheme for two-phase flow problems are discussed in [47, 69, 84, 127, 128]. Multicomponent fluid flow is simulated by DG method in [68].
- For level set simulations, a direct DG method for solving the Hamilton-Jacobi equations is designed in [24]. A fast sweeping solver based on this method to achieve rapid steady state convergence for eikonal equations is developed in [94].
- Even though not directly related to CFD, the DG algorithm developed in [23] can solve Boltzmann type equations for dimensions up to 5 (two space dimensions plus 3 phase space dimensions, plus time) with good resolution relative to Monte Carlo results on a single PC using a coarse mesh. Comparing with the WENO algorithm for the same purpose, the DG method is potentially more flexible in adaptivity. Again, it can be expected that this algorithm would be useful for rarefied gas simulation and other situations when a Boltzmann formulation of the physical process is necessary.

6 Concluding remarks

We have reviewed finite difference and finite volume WENO schemes, both in Eulerian and in Lagrangian framework, and discontinuous Galerkin finite element methods, for applications in computational fluid dynamics, especially for problems containing both shocks or high gradient regions and complex smooth region structures. Comparison among these methods in terms of applicability, efficiency and performance has been given. Time discretization issues have also been discussed.

A short summary can be given as follows. If the geometry is simple and a smooth curvilinear mesh can be used, then the high order finite difference WENO scheme would be the top choice, because of its simplicity and efficiency for multi-dimensional calculations. If the geometry is complicated, and the mesh is unstructured or non-smooth because of adaptivity or other considerations, then one would want to use either the finite volume WENO scheme or the discontinuous Galerkin method. The finite volume WENO scheme is more robust in the presence of strong shocks, while the discontinuous Galerkin method is less complicated for unstructured, distorted meshes. The memory requirement for the discontinuous Galerkin method is however much larger than that of the finite volume WENO scheme on the same mesh.

For future research in these methods, we could mention the following. For the WENO schemes, efficient and robust implicit time discretization involving WENO spatial discretizations is worth further development. It would be nice to study different designs of nonlinear weights for specific purposes, such as to enhance accuracy or to speed up steady state convergence. The further study of Hermite type WENO schemes, which would form a class of algorithms between finite volume schemes and discontinuous Galerkin method, is a good research area. For the discontinuous Galerkin method, efficient time stepping techniques, especially for PDEs containing higher (than first) derivatives such as the Navier-Stokes equations, deserve more investigation. A posteriori error estimates and other tools to guide h - p adaptivity, so that the advantage of the DG method can be fully utilized, require more study.

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